Limits

Definitions

Precise Definition : We say $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x\to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x\to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x\to\infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x\to-\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x\to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x\to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$\lim_{x\to a} f(x) = L \;\Rightarrow\; \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ $\qquad$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \;\Rightarrow\; \lim_{x\to a} f(x) = L$

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x) \;\Rightarrow\; \lim_{x\to a} f(x)$ Does Not Exist

Properties

Assume $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and c is any number then,

1. $\lim_{x\to a}\left[c f(x)\right] = c\lim_{x\to a} f(x)$
2. $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\left[f(x) g(x)\right] = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^-} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a x^n + \cdots + b x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a x^n + \cdots + b x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a x^n + \cdots + c x + d = -\operatorname{sgn}(a)\infty$

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x\to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

Factor and Cancel

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$
$$= \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$
$$= \lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}$$
$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$ factor largest power of x out of both $p(x)$ and $q(x)$ and then compute limit.

$$\lim_{x\to-\infty}\frac{3x^2-4}{5x-2x^2} = \lim_{x\to-\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x\to-\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x\to-2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x<-2 \\ 1-3x & \text{if } x\ge -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x^2+5 = 9$$
$$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} 1-3x = 7$$

One sided limits are different so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to-2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \ge 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \ne \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \ne \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.