

Limits

Definitions

Precise Definition : We say $\lim_{x \rightarrow a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

“Working” Definition : We say $\lim_{x \rightarrow a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim_{x \rightarrow a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim_{x \rightarrow a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim_{x \rightarrow \infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim_{x \rightarrow -\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim_{x \rightarrow a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim_{x \rightarrow a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$$\lim_{x \rightarrow a} f(x) = L \Rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \quad \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = L \Rightarrow \lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x) \Rightarrow \lim_{x \rightarrow a} f(x) \text{ Does Not Exist}$$

Properties

Assume $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ both exist and c is any number then,

- $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$
- $\lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$

Basic Limit Evaluations at $\pm \infty$

Note : $\text{sgn}(a) = 1$ if $a > 0$ and $\text{sgn}(a) = -1$ if $a < 0$.

- $\lim_{x \rightarrow \infty} e^x = \infty$ & $\lim_{x \rightarrow -\infty} e^x = 0$
- $\lim_{x \rightarrow \infty} \ln(x) = \infty$ & $\lim_{x \rightarrow 0^-} \ln(x) = -\infty$
- If $r > 0$ then $\lim_{x \rightarrow \infty} \frac{b}{x^r} = 0$
- If $r > 0$ and x^r is real for negative x then $\lim_{x \rightarrow -\infty} \frac{b}{x^r} = 0$
- n even : $\lim_{x \rightarrow \pm \infty} x^n = \infty$
- n odd : $\lim_{x \rightarrow \infty} x^n = \infty$ & $\lim_{x \rightarrow -\infty} x^n = -\infty$
- n even : $\lim_{x \rightarrow \pm \infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
- n odd : $\lim_{x \rightarrow \infty} ax^n + \dots + bx + c = \text{sgn}(a)\infty$
- n odd : $\lim_{x \rightarrow -\infty} ax^n + \dots + cx + d = -\text{sgn}(a)\infty$

Evaluation Techniques

Continuous Functions

If $f(x)$ is continuous at a then $\lim_{x \rightarrow a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x \rightarrow a} g(x) = b$ then

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(b)$$

Factor and Cancel

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{x^2 - 2x} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+6)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x+6}{x} = \frac{8}{2} = 4 \end{aligned}$$

Rationalize Numerator/Denominator

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} &= \lim_{x \rightarrow 9} \frac{3 - \sqrt{x}}{x^2 - 81} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}} \\ &= \lim_{x \rightarrow 9} \frac{9 - x}{(x^2 - 81)(3 + \sqrt{x})} = \lim_{x \rightarrow 9} \frac{-1}{(x+9)(3 + \sqrt{x})} \\ &= \frac{-1}{(18)(6)} = -\frac{1}{108} \end{aligned}$$

Combine Rational Expressions

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x - (x+h)}{x(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x(x+h)} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2} \end{aligned}$$

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

- | | |
|---|---|
| 1. Polynomials for all x . | 7. $\cos(x)$ and $\sin(x)$ for all x . |
| 2. Rational function, except for x 's that give division by zero. | 8. $\tan(x)$ and $\sec(x)$ provided |
| 3. $\sqrt[n]{x}$ (n odd) for all x . | $x \neq \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$ |
| 4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$. | 9. $\cot(x)$ and $\csc(x)$ provided |
| 5. e^x for all x . | $x \neq \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$ |
| 6. $\ln x$ for $x > 0$. | |

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$.

Then there exists a number c such that $a < c < b$ and $f(c) = M$.

L'Hospital's Rule

If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute

$\lim_{x \rightarrow \pm\infty} \frac{p(x)}{q(x)}$ factor largest power of x out of both

$p(x)$ and $q(x)$ and then compute limit.

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \rightarrow \infty} \frac{x^2 \left(3 - \frac{4}{x^2}\right)}{x^2 \left(\frac{5}{x} - 2\right)} = \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}$$

Piecewise Function

$$\lim_{x \rightarrow -2} g(x) \text{ where } g(x) = \begin{cases} x^2 + 5 & \text{if } x < -2 \\ 1 - 3x & \text{if } x \geq -2 \end{cases}$$

Compute two one sided limits,

$$\lim_{x \rightarrow -2^-} g(x) = \lim_{x \rightarrow -2^-} x^2 + 5 = 9$$

$$\lim_{x \rightarrow -2^+} g(x) = \lim_{x \rightarrow -2^+} 1 - 3x = 7$$

One sided limits are different so $\lim_{x \rightarrow -2} g(x)$

doesn't exist. If the two one sided limits had been equal then $\lim_{x \rightarrow -2} g(x)$ would have existed and had the same value.