Derivatives

Definition and Notation

If \( y = f(x) \) then the derivative is defined to be \( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \).

If \( y = f(x) \) then all of the following are equivalent notations for the derivative.

\[
\begin{align*}
    f'(x) &= \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) \\
    f'(a) &= \left. \frac{dy}{dx} \right|_{x=a} = \frac{dy}{dx}_{x=a} = Df(a)
\end{align*}
\]

Interpretation of the Derivative

If \( y = f(x) \) then,

1. \( m = f'(a) \) is the slope of the tangent line to \( y = f(x) \) at \( x = a \) and the equation of the tangent line at \( x = a \) is given by \( y = f(a) + f'(a)(x-a) \).

Basic Properties and Formulas

If \( f(x) \) and \( g(x) \) are differentiable functions, \( c \) and \( n \) are any real numbers,

1. \( (c f)' = c f'(x) \)
2. \( (f \pm g)' = f' \pm g' \)
3. \( (f g)' = f' g + f g' \) - Product Rule
4. \( \left( \frac{f}{g} \right)' = \frac{f' g - f g'}{g^2} \) - Quotient Rule

Common Derivatives

\[
\begin{align*}
    \frac{d}{dx}(x) &= 1 \\
    \frac{d}{dx}(\sin x) &= \cos x \\
    \frac{d}{dx}(\cos x) &= -\sin x \\
    \frac{d}{dx}(\tan x) &= \sec^2 x \\
    \frac{d}{dx}(\sec x) &= \sec x \tan x \\
    \frac{d}{dx}(\cot x) &= -\csc x \cot x \\
    \frac{d}{dx}(\csc x) &= -\csc x \cot x \\
    \frac{d}{dx}(\log_a x) &= \frac{1}{x \ln a} \\
    \frac{d}{dx}(\ln x) &= \frac{1}{x} \\
    \frac{d}{dx}(e^x) &= e^x \\
    \frac{d}{dx}(a^x) &= a^x \ln(a) \\
    \frac{d}{dx}(\ln(\ln x)) &= \frac{1}{x \ln x} \\
    \frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} \\
    \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} \\
    \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{1+x^2}
\end{align*}
\]

Chain Rule Variants

The chain rule applied to some specific functions.

1. \( \frac{d}{dx}[(f(x))^n] = n[f(x)]^{n-1} f'(x) \)
2. \( \frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)} \)
3. \( \frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)} \)
4. \( \frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)] \)
5. \( \frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)] \)
6. \( \frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)] \)
7. \( \frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)] \)
8. \( \frac{d}{dx}(\cot[f(x)]) = -f'(x)\csc^2[f(x)] \)

Higher Order Derivatives

The second derivative is denoted as \( f''(x) = f''(x) = \frac{d^2 f}{dx^2} \) and is defined as the derivative of the first derivative, \( f''(x) \).

Implicit Differentiation

Find \( y' \) if \( e^{2x+y^2} + x^2y^2 = \sin(y) + 11x \). Remember \( y = y(x) \) here, so products/quotients of \( x \) and \( y \) will use the product/quotient rule and derivatives of \( y \) will the chain rule. The “trick” is to differentiate as normal and every time you differentiate a \( y \) you tack on a \( y' \) (from the chain rule). After differentiating solve for \( y' \).

\[
\begin{align*}
    e^{2x+y^2} + x^2y^2 &= \sin(y) + 11x \\
    2e^{2x+y^2} \frac{dy}{dx} + 2xy^2 + 2x^2y y' &= \cos(y) y' + 11 \\
    (2x^2y - 9e^{2x+y^2} - \cos(y)) y' &= 11 - 2e^{2x+y^2} - 3x^2y^2
\end{align*}
\]

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

\( x = c \) is a critical point of \( f(x) \) provided either

1. \( f'(c) = 0 \)  or  \( f''(c) \) doesn't exist.

Increasing/Decreasing

1. If \( f''(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is increasing on the interval \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is decreasing on the interval \( I \).
3. If \( f''(x) = 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is constant on the interval \( I \).

Concave Up/Concave Down

1. If \( f''(x) > 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave up on the interval \( I \).
2. If \( f''(x) < 0 \) for all \( x \) in an interval \( I \) then \( f(x) \) is concave down on the interval \( I \).

Inflection Points

\( x = c \) is an inflection point of \( f(x) \) if the concavity changes at \( x = c \).
Extrema

Relative (local) Extrema
1. \( x = c \) is a relative (or local) maximum of \( f(x) \) if \( f(c) \geq f(x) \) for all \( x \) near \( c \).
2. \( x = c \) is a relative (or local) minimum of \( f(x) \) if \( f(c) \leq f(x) \) for all \( x \) near \( c \).

1st Derivative Test
If \( x = c \) is a critical point of \( f(x) \), then \( x = c \) is a relative extremum of \( f(x) \) if:
1. \( f'(c) > 0 \) to the left of \( x = c \) and \( f'(x) < 0 \) to the right of \( x = c \).
2. \( f'(c) < 0 \) to the left of \( x = c \) and \( f'(x) > 0 \) to the right of \( x = c \).
3. not a relative extremum of \( f(x) \) if \( f'(c) \) is the same sign on both sides of \( x = c \).

2nd Derivative Test
If \( x = c \) is a critical point of \( f(x) \) such that \( f''(c) = 0 \), then:
1. if \( f''(c) > 0 \), \( x = c \) is a relative minimum.
2. if \( f''(c) < 0 \), \( x = c \) is a relative maximum.
3. if \( f''(c) = 0 \), test the second derivative test on each critical point.

Finding Relative Extrema and/or Classify Critical Points
1. Find all critical points of \( f(x) \).
2. Use the 1st derivative test or the 2nd derivative test on each critical point.

Mean Value Theorem
If \( f(x) \) is continuous on the closed interval \([a,b]\) and differentiable on the open interval \((a,b)\) then there exists a \( c \) such that \( f'(c) = \frac{f(b) - f(a)}{b-a} \).

Newton’s Method
If \( x_n \) is the \( n \)th guess for the root/solution of \( f(x) = 0 \), then \((n+1)\)th guess is \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) provided \( f'(x_n) \) exists.

Optimization

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at \( 4 \) ft/sec. How fast is the top moving after 12 sec?

\[
\frac{dx}{dt} = 4 \quad \frac{dy}{dt} = \text{unknown}
\]

By the 2nd derivative test this is a rel. max. and so is the answer we're after. Finally, find:
\[
x = 500 - 2(125) = 250
\]
The dimensions are then 250 x 125.

Ex. We’re enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.

\[
\begin{align*}
\text{Building:} & \quad y \quad \text{Building} \\
\text{Yard:} & \quad x \quad \text{Yard}
\end{align*}
\]

Maximize \( A = xy \) subject to constraint of \( x + 2y = 500 \). Solve constraint for \( x \) and plug into area.

\[
x = 500 - 2y \quad A = (500 - 2y)2 = 500y - 2y^2
\]

Differentiate and find critical point(s).

\[
A' = 500 - 4y \quad y = 125
\]

By 2nd deriv. test this is a rel. max. and so is the answer we’re after. Finally, find:

\[
x = 500 - 2(125) = 250
\]

The dimensions are then 250 x 125.

Ex. Determine point(s) on \( y = x^2 + 1 \) that are closest to \((0,2)\).

Minimize \( f = d^2 = (x - 0)^2 + (y - 2)^2 \) and the constraint is \( y = x^2 + 1 \). Solve constraint for \( x^2 \) and plug into the function.

\[
x^2 = y - 1 \quad f = x^3 + (y - 2)^2 = y - 1 + (y - 2)^2 = y^2 - 3y + 3
\]

Differentiate and find critical point(s).

\[
f' = 2y - 3 = 0 \quad y = \frac{3}{2}
\]

The 2 points are then \((\frac{3}{2},\frac{17}{4})\) and \((\frac{1}{2},\frac{1}{4})\)